hep-th/9609157
IPM-96-165
Sep 1996

On the Picard-Fuchs equations of the SW models

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Abstract

We obtain the closed form of the Picard-Fuchs equations for $N = 2$ supersymmetric Yang-Mills theories with classical Lie gauge groups. For a gauge group of rank r, there are $r - 1$ regular and an exceptional differential equations. We describe the series solutions of the Picard-Fuchs equations in the semi-classical regime.

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From duality and holomorphy, Seiberg and Witten[1] have obtained the exact prepotential of $N = 2$ SYM theory with gauge group $SU(2)$ by studying the singularities of its moduli space at strong coupling. For a gauge group of rank r, the Seiberg and Witten's data is a hyperelliptic curve with r complex dimensional moduli space with certain singulareties and a meromorphic one form, (E_{u_i}, λ_{SW}). More precisely, the prepotential of $N = 2$ SYM in the Coulomb phase can be described with the aid of a family of complex curves with the identification of the v.e.v.'s a_i and their duals a_i^D with the periods of the curve

$$a_i = \oint_{\alpha_i} \lambda_{SW} \quad \text{and} \quad a_i^D = \oint_{\beta_i} \lambda_{SW}, \tag{1}$$

where α_i and β_i are the homology cycles of the corresponding Riemann surface.

To find the periods one should calculate the above integrals, or one may use the fact that the periods $\Pi = (a_i, a_i^D)$ satisfy the Picard-Fuchs equations. So it is important to find the Picard-Fuchs equations which also help in the instanton calculus.

In this letter we obtain closed forms for the Picard-Fuchs equations for classical Lie gauge groups. Recently, some of these equations have been obtained in [2] and [3].

The Seiberg-Witten's data (E_{u_i}, λ_{SW}) for classical gauge groups are known [4] and [5], and take the following form

$$\begin{aligned} y^2 &= W^2(x) - \Lambda^{2\hat{h}} x^{2k} \\ \lambda_{SW} &= (kW - x\frac{dW}{dx})\frac{dx}{y} \end{aligned} \tag{2}$$

where $\hat{h}$ is the dual Coxeter number of the Lie gauge group and

$$W(x) = x^m - \sum_{i=2}^{m} u_i x^{m-i} \tag{3}$$

with $m = r + 1, i = 2, 3, ..., r + 1$ for A_r series and $m = 2r, i = 2, 4, ..., 2r$ for B_r, C_r, D_r series, and u_i's, the Casimirs of the gauge groups. Also $k = m - \hat{h}$. Note that the D_r series has an exceptional Casimir, t, of degree r, but in our notation we set $u_{2r} = t^2$.

From explicit form of λ_{SW} and the fact that the λ_{SW} is lineary dependent on Casimirs, setting $\frac{\partial}{\partial u_i} = \partial_i$ we have

$$\begin{aligned} \partial_i \lambda_{SW} &= -\frac{x^{m-i}}{y} dx + d(*), \\ \partial_i \partial_j \lambda_{SW} &= -\frac{x^{2m-i-j}}{y^3} W(x) dx + d(*). \end{aligned} \tag{4}$$

By direct calculation one can see that

$$\frac{d}{dx}(\frac{x^n}{y}) = (n-k)\frac{x^{n-1}}{y} + (\frac{kx^{n-1}W - x^n \frac{dW}{dx}}{y^3})W. \tag{5}$$

By inserting equation (3) in (5) we have

$$\frac{d}{dx}(\frac{x^n}{y}) = (n-k)\frac{x^{n-1}}{y} - \hat{h}\frac{x^{m+n-1}}{y^3}W + \sum_{i=2}^{m}(m-k-i)u_i\frac{x^{m+n-1-i}}{y^3}W \qquad (6)$$

Now from equations (4) we can find the second order differential equation for the periods Π as follows

$$\mathcal{L}_n = (k-n)\partial_{m-n+1} + \hat{h}\partial_2\partial_{m-n-1} - \sum_{i=2}^{m}(m-k-i)u_i\partial_i\partial_{m-n+1}. \qquad (7)$$

where $n = s-1$ for A_r series and $n = 2s-1$ for B_r, C_r and D_r series and $s = 1, ..., r-1$. Note that in the A_r series, for $s = 1$, the above expression is not valid. In fact we should be careful in the final step in the derivation of the equation (7), that $\mathcal{L}_0^{A_r}$ is

$$\mathcal{L}_0^{A_r} = (r+1)\partial_2\partial_r - \sum_{i=2}^{r}(r+1-i)u_i\partial_{i+1}\partial_{r+1} \qquad (8)$$

Moreover for $s > r-1$ equation (6) does not give the second order differential equation with respect to u_i. So, by this method we can only find $r-1$ equations which we call *regular* equations. Also from the equation (4) we have the following identity

$$\mathcal{L}_{i,j;p,q} = \partial_i\partial_j - \partial_p\partial_q, \quad i+j = p+q \qquad (9)$$

The rth equation, the *exceptional* equation, can be obtain from the following linear combination

$$D = (k-m)d(\frac{x^{m+1}}{y}) + \sum_{i=2}^{m}(m-k+i)u_i d(\frac{x^{m+1-i}}{y}) \qquad (10)$$

or

$$D = \lambda_{SW} - (\sum_{i=2}^{m} i(i-2)u_i\frac{x^{m-i}}{y} + \sum_{j,i=2}^{m} iju_iu_j\frac{x^{2m-i-j}}{y^3}W - \hat{h}^2\Lambda^{2\hat{h}}\frac{x^{2k}}{y^3}W)dx. \qquad (11)$$

So from the equation (4), the *exceptional* differential equation for the periods Π are

$$\mathcal{L}_r = 1 + \sum_{i=2}^{m} i(i-2)u_i\partial_i + \sum_{j,i=2}^{m} iju_iu_j\partial_i\partial_j - \hat{h}^2\Lambda^{2\hat{h}}\partial_{\hat{h}}^2. \qquad (12)$$

for A_r, D_r and C_r with odd r. For B_r and C_r with even r the last term should be changed to $-\hat{h}^2\Lambda^{2\hat{h}}\partial_{\hat{h}-1}\partial_{\hat{h}+1}$.

This equation together with the equations (7) and (9) give a complete set of the Picard-Fuchs equations for the periods (a_i, a_i^D).

To study the series solutions of the Picard-Fuchs equations in the semi-classical regime, let us rewrite the Picard-Fuchs equations in terms of Euler derivative $\vartheta_i = u_i\partial_i$. The regular equations for $s \neq r-1$ become

$$\mathcal{L}_n = [(k+1-n)\vartheta_{m-n+1} - 1]\frac{\vartheta_{m-n+1}}{u_{m-n+1}} - \sum_{i\neq m-n+1, i=2}^{m}\frac{(m-k-i)}{u_{m-n+1}}\vartheta_i\vartheta_{m-n+1}$$

$$+\frac{\hat{h}}{u_2 u_{m-n-1}}\vartheta_2\vartheta_{m-n-1} \qquad (13)$$

If $s=r-1$ the last term should be replaced by $\frac{\hat{h}}{u_2^2}\vartheta_2(\vartheta_2-1)$. Again, we should be careful for $s=1$ in the A_r series. From equation (8) we have

$$\mathcal{L}_0^{A_r}=\frac{(r+1)}{u_2 u_r}\vartheta_2\vartheta_r-\frac{u_r}{u_{r+1}^2}\vartheta_{r+1}(\vartheta_{r+1}-1)-\sum_{i=2}^{r-1}\frac{(r+1-i)u_i}{u_{i+1}u_{r+1}}\vartheta_{i+1}\vartheta_{r+1} \qquad (14)$$

The exceptoinal equation changes to

$$\mathcal{L}_r=(1-\sum_{i=2}^{m}i\vartheta_i)^2-\frac{\hat{h}^2\Lambda^{2\hat{h}}}{u_{\hat{h}^2}}\vartheta_{\hat{h}}(\vartheta_{\hat{h}}-1) \qquad (15)$$

for A_r, D_r and C_r with odd r, and

$$\mathcal{L}_r=(1-\sum_{i=2}^{m}i\vartheta_i)^2-\frac{\hat{h}^2\Lambda^{2\hat{h}}}{u_{\hat{h}-1}u_{\hat{h}+1}}\vartheta_{\hat{h}-1}\vartheta_{\hat{h}+1} \qquad (16)$$

for B_r and C_r with even r.

Let us define the variables x_s

$$\begin{aligned} x_s &= \tfrac{u_{m-n+1}}{u_2 u_{m-n-1}} \quad s=1,...,r-1 \\ x_r &= \tfrac{\Lambda^{2\hat{h}}}{u_{\hat{h}}^2} \qquad (or\ \ x_r=\tfrac{\Lambda^{2\hat{h}}}{u_{\hat{h}-1}u_{\hat{h}+1}}) \end{aligned} \qquad (17)$$

for B_r, C_r and D_r corresponding to above, and the variables

$$\begin{aligned} x_s &= \tfrac{u_{m-n+1}}{u_2 u_{m-n-1}} \quad s=2,...,r-1 \\ x_1 &= \tfrac{\Lambda^{2(r+1)}}{u_2 u_r^2} \qquad x_r=\tfrac{\Lambda^{2(r+1)}}{u_{(r+1)}^2}. \end{aligned} \qquad (18)$$

for the A_r series. We can construct the power series solution of the Picard-Fuchs equations around $(x_i)=(0)$ [2]

$$\omega(a_1,...,a_r;x_1,...,x_r)=\sum_{l_1,...,l_r=0}C_{l_1,...,l_r}x_1^{l_1+a_1}...x_r^{l_r+a_r} \qquad (19)$$

Let us take $\alpha_i(l_j)=\alpha_i(a_1,...,a_r;l_1,...,l_r)$ be the power of u_i when equation (19) is reexpressed in term of u_i's. By inserting ω in the Picard-Fuchs equations, one can obtain the indicial and recursion relations. For example the indicial relation for B_r, C_r and D_r are

$$[(k+1-n)\alpha_{m-n+1}(0)-1]\alpha_{m-n+1}(0) \quad - \quad \sum_{i\neq m-n+1,i=2}^{m}(m-k-i)\alpha_i(0)\alpha_{m-n+1}(0)=0$$

$$(1-\sum_{i=2}^{m} i\alpha_i(0))^2 \quad = \quad 0 \tag{20}$$

Also the recursion relations are

$$\begin{aligned}
C_{l_1,\ldots,l_r} &= \tfrac{-\hat{h}\alpha_2(l_s-1)\alpha_{m-n-1}(l_s-1)}{\Delta_s} C_{l_1,\ldots,l_s-1,\ldots,l_r} \qquad i=1,\ldots,r-2 \\
C_{l_1,\ldots,l_r} &= \tfrac{-\hat{h}\alpha_2(l_{r-1}-1)(\alpha_2(l_{r-1}-1)-1)}{\Delta_s} C_{l_1,\ldots,l_{r-1}-1,l_r} \\
C_{l_1,\ldots,l_r} &= \tfrac{\hat{h}^2\alpha_{\hat{h}}(l_r-1)(\alpha_{\hat{h}}(l_r-1)-1)}{(1-\sum_i i\alpha_i(l_j))^2} C_{l_1,\ldots,l_r-1}
\end{aligned} \tag{21}$$

where $\alpha_i(0)$ means that all $l_j=0$ and $\alpha_i(l_s-1)$ means that the sth l should be set equal to l_s-1 and other l's are fixed, also

$$\Delta_s = [(k+1-n)\alpha_{m-n+1}(l_j)-1]\alpha_{m-n+1}(l_j) - \sum_{i\neq m-n+1, i=2}^{m} (m-k-i)\alpha_i(l_j)\alpha_{m-n+1}(l_j) \tag{22}$$

Note that in the last relation of the equation (21) the suitable change as noted above should be made. By the same method one can obtain the indicial and recursion relations for the A_r series. This method can be applied for the $N=2$ theories with massless hypermultiplets which their curves are in the form of (2).

After completion of this work, i recived the paper [6] which is paid to the same problem.

I would like to thank M. Khorrami for helpful discussions.

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